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The term 'programming' means 'planning' and it refers to a particular plan of action from amongst several alternatives for maximising or minimising a function under given restrictions such as maximising profit or minimising cost, etc. The term 'linear' means that 'all inequations' or 'equations used' and the function to be maximised or minimised are linear. Thus, linear programming is a technique for resource utilisation.

LINEAR PROGRAMMING

LINEAR INEQUALITIES

An inequality or inequation is said to be linear, if each variable occurs in first degree only and there is no term involving the product of the variables.

e.g. $ax + b \leq 0$, $ax + by + c > 0$, $ax \leq 4$, etc.

(i) **Linear Inequality in One Variable** A linear inequality or inequation, which has only one variable, is called linear inequality or inequation in one variable. **e.g.** $ax + b < 0$, where $a \neq 0$.

(ii) **Linear Inequality in Two Variables** A linear inequality, which have only two variables, is called linear inequality in two variables.

e.g. $3x + 11y \leq 0$, $4t + 3s > 0$.

CHAPTER CHECKLIST

- Linear Inequalities
- Linear Programming Problem (LPP)

Solution of a Linear Inequality in Two Variables by Graphical Method

Suppose, given linear inequality is $ax + by \leq c$ or $ax + by \geq c$ or $ax + by < c$ or $ax + by > c$, then to find its solution by graphical method, we use the following steps

I. Consider the equation $ax + by = c$ in place of given inequality, which represents a straight line in XY-plane.

II. Put $x = 0$ in the equation obtained in step I to get the point, where the line meets Y-axis. Similarly, put $y = 0$ to obtain a point, where the line meets X-axis.

III. Draw a line joining the points obtained in step II. If the inequality is of the form $<$ or $>$, then draw dotted line to indicate that the points on the line are excluded from the solution set. Otherwise, mark it by thick or dark line to indicate that the points on this line are included in the solution set.

IV. Take any point {preferable origin, i.e. (0, 0)} not lying on the line and check whether this satisfies the given linear inequality or not.

V. If the inequality is satisfied, then shade that portion of the plane, which contains the chosen point. Otherwise, shade that portion, which does not contain the chosen point.

Thus, shaded region obtained in step V, represents the required solution set.

EXAMPLE |1| Solve the inequality $y + 8 \geq 2x$ by graphical method.

Sol. Given inequality is $y + 8 \geq 2x$.

In equation form, it can be written as $y + 8 = 2x$

or $2x - y = 8$ ...(i)

On putting $y = 0$ in Eq. (i), we get

$$2x - 0 = 8 \Rightarrow x = 4$$

$\therefore$ Line (i) cuts the X-axis at $A(4, 0)$.

On putting $x = 0$ in Eq. (i), we get

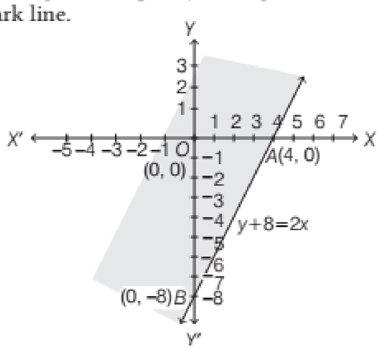
$$2 \times 0 - y = 8$$

$$\Rightarrow \quad y = -8$$

$\therefore$ Line (i) cuts the Y-axis at $B(0, -8)$.

On plotting $A(4, 0)$ and $B(0, -8)$ on graph paper and then joining them, we get the line AB.

Since, given inequality has sign '$\geq$', so we draw dark line.



Now, putting $x = y = 0$ in $y + 8 \geq 2x$, we get

$$0 + 8 \geq 2(0) \Rightarrow 8 \geq 0, \text{ which is true.}$$

So, the region containing (0, 0) is the region above the line AB.

Hence, the shaded region containing origin, is the required solution set.

LINEAR PROGRAMMING PROBLEM (LPP)

A linear programming problem is one that is concerned with finding the optimal value (maximum or minimum value) of a linear function of several variables, subject to the constraints that the variables are non-negative and satisfy a set of linear inequalities.

Mathematical Form of LPP

The general mathematical form of a linear programming problem may be written as follows

Maximise or Minimise, $Z = c_1 x + c_2 y$, subject to constraints are $a_1 x + b_1 y \leq d_1$, $a_2 x + b_2 y \leq d_2$, etc., and non-negative restrictions are $x \geq 0$, $y \geq 0$.

Important Terms Related to LPP

There are various terms related to a linear programming problem. Definition of these terms are given below

(i) **Constraints** The linear inequations or inequalities or restrictions on the variables of a linear programming problem are called constraints. The conditions $x \geq 0$, $y \geq 0$ are called **non-negative restrictions**.

(ii) **Optimisation Problem** A problem which seeks to maximise or minimise a linear function subject to certain constraints as determined by a set of linear inequalities is called an optimisation problem. Linear programming problems are special type of optimisation problems.

(iii) **Objective Function** A linear function of two or more variables which has to be maximised or minimised under the given restrictions is called an objective function. The variables used in the objective function are called **decision variables**.

(iv) **Optimal Value** The maximum or minimum value of an objective function is known as the optimal value of LPP.

(v) **Feasible and Infeasible Region** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the feasible region or solution region. Each point in this region represents a feasible choice. The region other than feasible region is called an infeasible region.

(vi) **Bounded and Unbounded Region** A feasible region of a system of linear inequalities is said to be bounded, if it can be enclosed within a circle.

Otherwise, it is said to be unbounded region, i.e. the feasible region does extend infinitely in any direction.

(vii) **Feasible and Infeasible Solution** Points within and on the boundary of the feasible region, represents feasible solution of the constraints. Any point outside the feasible region is called an infeasible solution.

(viii) **Optimal Feasible Solution** A feasible solution at which the objective function has optimal value (maximum or minimum), is called the optimal solution or optimal feasible solution of the linear programming problem.

(ix) **Optimisation Technique** The process of obtaining the optimal solution of the linear programming problem is called optimisation technique.

Some Important Theorems

We know that every point in the feasible region will satisfy all the constraints given in the LPP and non-negative restrictions. Also, a feasible region contains infinitely many points. So, it is not easy to find a point that gives a maximum or minimum value of objective function. To handle this situation, we use the following theorems:

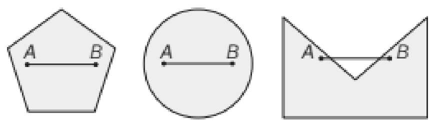
Theorem 1 Let R be the feasible region (convex polygon) for a linear programming problem and $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region. (A corner point of a feasible region is a point of intersection of two boundary lines in the region)

Theorem 2 Let R be the feasible region for a linear programming problem and $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and minimum value on R and each of these occurs at a corner point (vertex) of R.

Note

(i) If R is unbounded, then a maximum or minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R.

(ii) A polygon is said to be convex polygon, if the line joining any two of its points lies completely in the region. Figures (i) and (ii) are convex polygon but Figure (iii) is not a convex polygon.



(i) (ii) (iii)

Graphical Method for Solving Linear Programming Problem (LPP)

The graphical method is suitable for solving linear programming problems containing two variables only. This method of solving linear programming problem is referred as corner point method. The process of this method is as follows

1. Find the feasible region of the LPP and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Find the value of objective function $Z = ax + by$ at each corner point. Let M and m respectively, denote the largest and the smallest values at these points.

(i) When the feasible region is **bounded**, then M and m are the maximum and minimum values of Z.

(ii) When the feasible region is **unbounded**, then

(a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

Note

(i) A half plane (if a line divides XY-plane into two parts, then each part is known as half plane) in XY-plane is called an open half plane, if the line separating the plane is not included in the half plane.

(ii) It two corner points of the feasible region are both optimal solution of the same type, i.e. both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

EXAMPLE |2| Solve the following linear programming problem graphically.

Maximise $Z = 34x + 45y$

subject to constraints $x + y \le 300$, $2x + 3y \le 70$; $x, y \ge 0$

[All India 2017]

Sol. We have the following LPP

Maximise $Z = 34x + 45y$

subject to the constraints

$x + y \le 300,\ 2x + 3y \le 70;\ x, y \ge 0$

Now, considering the inequations as equations, we get

$x + y = 300$ ...(i)

and $2x + 3y = 70$ ...(ii)

Table for line $x + y = 300$ is

x	0	300
y	300	0

So, the line passes through the points (0, 300) and (300, 0).

On putting (0, 0) in the inequality $x + y \le 300$, we get

$0 + 0 \le 300$, which is true.

So, the half plane is towards the origin.

Table for line $2x + 3y = 70$ is

x	35	0
y	0	70/3

So, the line passes through the points (35, 0) and (0, 70/3).

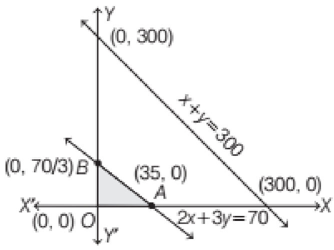
On putting (0, 0) in the inequality $2x + 3y \le 70$, we get

$0 + 0 \le 70$, which is true.

So, the half plane is towards the origin.

Also, $x, y \ge 0$, so the region lies in the 1st quadrant.

The graphical representation of the system of inequations is as given below.



Clearly, the feasible region is $OABO$, where the corner points are $O(0,0)$, $A(35,0)$ and $B(0,70/3)$.

Now, the values of Z at corner points are as follow

Corner points	$Z = 34x + 45y$
$O(0,0)$	$Z = 0 + 0 = 0$
$A(35,0)$	$Z = 34 \times 35 + 45 \times 0 = 1190$
$B(0,70/3)$	$Z = 34 \times 0 + 45 \times \frac{70}{3} = 1050$

Hence, the maximum value of Z is 1190 at (35, 0).

EXAMPLE |3| Solve the following linear programming problems graphically.

Maximise $Z = 2x + 3y$, subject to constraints
$x + 2y \le 10$, $2x + y \le 14$ and $x \ge 0$, $y \ge 0$

Sol. Given, maximise $Z = 2x + 3y$

subject to constraints

$x + 2y \le 10$ ...(i)

$2x + y \le 14$ ...(ii)

and $x \ge 0$, $y \ge 0$ ...(iii)

Shade the region to the right of Y-axis to show $x \ge 0$ and above X-axis to show $y \ge 0$.

Table for line $x + 2y = 10$ is

x	0	4	10
y	5	3	0

So, the line is passing through the points (0, 5), (4, 3) and (10, 0).

On putting (0, 0) in the inequality $x + 2y \le 10$, we get $0 + 0 \le 10$, which is true.

So, the half plane is towards the origin.

Table for line $2x + y = 14$ is

x	4	6	7
y	6	2	0

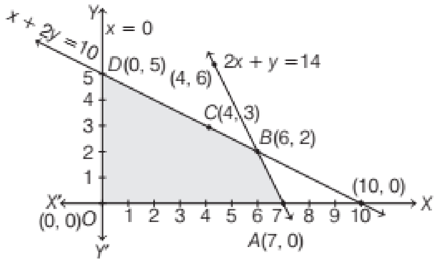
So, the line is passing through the points (4, 6), (6, 2) and (7, 0).

On putting (0, 0) in the inequality $2x + y \le 14$, we get $0 + 0 \le 14$, which is true.

So, the half plane is towards the origin.

The intersection point of lines corresponding to Eqs. (i) and (ii) is $B(6, 2)$.

On shading the common region, we get the feasible region $OABD$.



The values of Z at corner points are given below

Corner points	$Z = 2x + 3y$
$O(0, 0)$	$Z = 2 \times 0 + 3 \times 0 = 0$
$A(7, 0)$	$Z = 2 \times 7 + 3 \times 0 = 14$
$B(6, 2)$	$Z = 2 \times 6 + 3 \times 2 = 18$
$D(0, 5)$	$Z = 2 \times 0 + 3 \times 5 = 15$

Hence, the maximum value of Z is 18 at the point $B(6, 2)$.

EXAMPLE |4| Solve the following linear programming problems graphically.

Minimise $Z = x + y$, subject to constraints
$3x + 2y \ge 12$, $x + 3y \ge 11$ and $x \ge 0$, $y \ge 0$

Sol. Given, minimise $Z = x + y$

subject to constraints

$3x + 2y \ge 12$ ...(i)

$x + 3y \ge 11$ ...(ii)

and $x \ge 0$, $y \ge 0$ ...(iii)

Table for line $3x + 2y = 12$ is

x	0	4
y	6	0

So, the line $3x + 2y = 12$ is passing through the points (0, 6) and (4, 0).

On putting (0, 0) in the inequality $3x + 2y \ge 12$, we get $0 + 0 \ge 12$, which is not true.

So, the half plane is away from the origin.

Table for line $x + 3y = 11$ is

x	2	11
y	3	0

So, the line $x + 3y = 11$ is passing through the points (2, 3) and (11, 0).

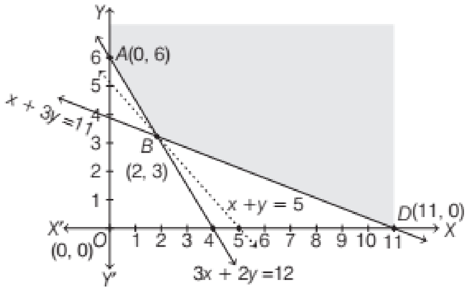
On putting (0, 0) in the inequality $x + 3y \ge 11$, we get
$0 + 0 \ge 11$,
which is not true.

So, the half plane is away from the origin.

Also, $x \ge 0$ and $y \ge 0$, so the region lies in I quadrant.

The intersection point of lines corresponding to Eqs. (i) and (ii) is $B(2, 3)$.

The shaded region ABD represents the graph of the given constraints. The corner points of the boundary of the graph ABD are $A(0, 6)$, $B(2, 3)$ and $D(11, 0)$.



The values of Z at corner points are given below

Corner points	$Z = x + y$
$A(0, 6)$	$Z = 0 + 6 = 6$
$B(2, 3)$	$Z = 2 + 3 = 5$
$D(11, 0)$	$Z = 11 + 0 = 11$

Thus, the minimum value of Z is 5 at the point $B(2, 3)$.

As the feasible region is unbounded, therefore 5 may or may not be the minimum value of Z. For this, we draw a dotted graph of the inequality $x + y < 5$ and check whether the resulting half plane has points in common with a feasible region or not. It can be seen that the feasible region has no common point with $x + y < 5$.

Therefore, the minimum value of Z is 5 at $B(2, 3)$.

EXAMPLE |5| Solve the following linear programming problems graphically.

Maximise $Z = x + y$, subject to constraints

$x - y \le -1$, $-x + y \le 0$ and $x, y \ge 0$

Sol. Our problem is to maximise

$$Z = x + y \quad \text{...(i)}$$

subject to constraints, $x - y \le -1$ ...(ii)

$-x + y \le 0$ ...(iii)

and $x \ge 0, y \ge 0$ ...(iv)

Table for line $x - y = -1$ is

x	0	−1
y	1	0

So, the line $x - y = -1$ is passing through the points (0, 1) and (− 1, 0).

On putting (0, 0) in the inequality $x - y \le -1$, we get

$0 - 0 \le -1$

$\Rightarrow$ $0 \le -1$, which is not true.

So, the half plane is away from the origin.

Table for line $-x + y = 0$ is

x	0	1
y	0	1

So, the line $-x + y = 0$ is passing through (0, 0) and (1, 1).

On putting (2, 0) in the inequality $-x + y \le 0$, we get

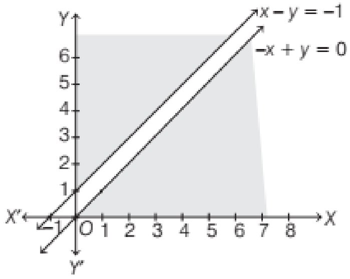
$-2 + 0 \le 0$

$\Rightarrow$ $-2 \le 0$, which is true.

So, the half plane is towards the X-axis.

Also, $x, y \ge 0$.

So, the region lies in the I quadrant.



From the above graph it is clearly shown that there is no common region.

Hence, there is no feasible region and thus Z has no maximum value.

Note If there is no common region, then we do not determine the minimum/maximum value.

EXAMPLE |6| Solve the following problem graphically.

Minimise and maximise $Z = 3x + 9y$,

subject to constraints

$x + 3y \le 60$, $x + y \ge 10$, $x \le y$; $x \ge 0$, $y \ge 0$ **[NCERT]**

Sol. Given objective function is maximise and minimise

$Z = 3x + 9y$

subject to constraints,

$x + 3y \le 60$ ...(i)

$x + y \ge 10$ ...(ii)

$x \le y$ ...(iii)

and $x \ge 0, y \ge 0$ ...(iv)

Table for line $x + 3y = 60$ is

x	60	0
y	0	20

So, the line $x + 3y = 60$ is passing through (60, 0) and (0, 20).

On putting (0, 0) in the inequality $x + 3y \le 60$, we get

$0 + 0 \le 60$, which is true.

So, the half plane is towards the origin.

Table for line $x + y = 10$ is

x	10	0
y	0	10

So, the line $x + y = 10$ is passing through (10, 0) and (0, 10).

On putting (0, 0) in the inequality $x + y \ge 10$, we get

$0 + 0 \ge 10$,

which is not true.

So, the half plane is away from the origin.

Table for line $x = y$ is

x	0	5
y	0	5

So, the line $x = y$ is passing through (0, 0) and (5, 5).

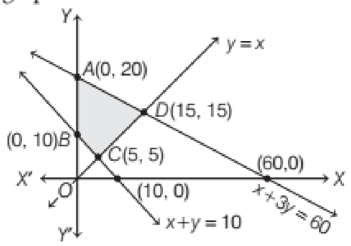
On putting (0, 4) is the inequality $x \le y$, we get

$0 \le 4$, which is true.

So, the half plane is towards the Y-axis.

Also, $x \ge 0$ and $y \ge 0$, so the region lies in the I quadrant.

On drawing these lines on graph paper, we get the following graph



The intersection point of lines corresponding to Eqs. (i) and (iii) is D (15, 15) corresponding to Eqs. (ii) and (iii) is C(5, 5).

Now, shading the common region, we get the feasible region $ABCDA$.

Hence, the region is bounded and corner points of feasible region are A(0, 20), B(0, 10), C(5, 5) and D(15, 15).

The values of Z at corner points are given below

Corner points	$Z = 3x + 9y$
A(0, 20)	$Z = 3(0) + 9(20) = 180$
B(0, 10)	$Z = 3(0) + 9(10) = 90$
C(5, 5)	$Z = 3(5) + 9(5) = 60$
D(15, 15)	$Z = 3(15) + 9(15) = 180$

Here, minimum value of Z is 60 which occur at point C(5, 5) and maximum value of Z is 180 which occur at two points A(0, 20) and D(15, 15). So, each point on the line segment AD will give the maximum value of Z.

EXAMPLE |7| Solve the following linear programming problems graphically.

Maximise $Z = -x + 2y$, subject to constraints are

$x \ge 3,\ x + y \ge 5,\ x + 2y \ge 6$ and $x, y \ge 0$.

Sol. Our problem is to maximise $Z = -x + 2y$ ...(i)

subject to constraints,

$x \ge 3$ ...(ii)

$x + y \ge 5$ ...(iii)

$x + 2y \ge 6$ ...(iv)

and $x \ge 0,\ y \ge 0$ ...(v)

Table for line $x + y = 5$ is

x	0	5
y	5	0

So, the line passes through the points (0, 5) and (5, 0).

On putting (0, 0) in the inequality $x + y \ge 5$, we get

$0 + 0 \ge 5$

$\Rightarrow$ $0 \ge 5$, which is not true.

So, the half plane is away from the origin.

Table for line $x + 2y = 6$ is

x	0	6
y	3	0

So, the line passes through the points (0,3) and (6,0).

On putting (0, 0) in the inequality $x + 2y \ge 6$, we get

$0 + 2 \times 0 \ge 6$

$\Rightarrow$ $0 \ge 6$, which is not true.

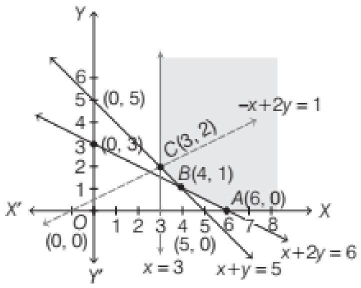
So, the half plane is away from the origin.

Also, $x \ge 3$, so the region is away from the orgin.

Since, $x \ge 0,\ y \ge 0$.

So, the region lies in the I quadrant.

The points of intersection of lines $x = 3$ and $x + y = 5$ is C(3, 2) and lines $x + 2y = 6$ and $x + y = 5$ is B(4, 1). It can be seen that the feasible region is unbounded.



The corner points of the feasible region are A(6, 0), B(4, 1) and C(3, 2).

The values of Z at corner points are given below

Corner points	$Z = -x + 2y$
A(6, 0)	$Z = -6 + 2 \times 0 = -6$
B(4, 1)	$Z = -4 + 2 \times 1 = -2$
C(3, 2)	$Z = -3 + 2 \times 2 = 1$

As the feasible region is unbounded, therefore $Z = 1$ may or may not be the maximum value. For this, we draw the graph of the inequality $-x + 2y > 1$ and check whether the resulting half plane has points in common with the feasible region or not. The feasible region have points in common with the $-x + 2y > 1$.

Therefore, $Z = 1$ is not the maximum value.

Hence, Z has no maximum value.

EXAMPLE |8| Solve the following LPP graphically.

Minimise $Z = 5x + 10y$

subject to the constraints

$x + 2y \le 120,\ x + y \ge 60,\ x - 2y \ge 0$ and $x, y \ge 0$

[Delhi 2017]

Sol. Our problem is to minimise

$$Z = 5x + 10y \quad \text{...(i)}$$

subject to constraints

$$x + 2y \le 120 \quad \text{...(ii)}$$
$$x + y \ge 60 \quad \text{...(iii)}$$
$$x - 2y \ge 0 \quad \text{...(iv)}$$

and $x \ge 0,\ y \ge 0$

Firstly, draw the graph of the line $x + 2y = 120$.

x	0	120
y	60	0

Put (0, 0) in the inequality $x + 2y \le 120$, we get

$0 + 2 \times 0 \le 120 \Rightarrow 0 \le 120$, which is true.

So, the half plane is towards the origin. Secondly, draw the graph of the line $x + y = 60$.

x	0	60
y	60	0

Put (0, 0) in the inequality $x + y \ge 60$, we get

$0 + 0 \ge 60 \Rightarrow 0 \ge 60$, which is false.

So, the half plane is away from the origin.

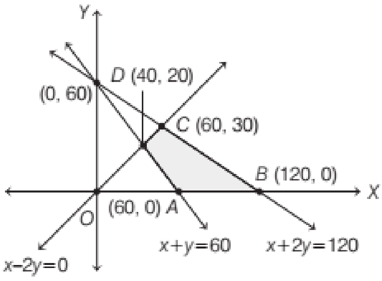
Thirdly, draw the graph of the line $x - 2y = 0$.

x	0	10
y	0	5

Put (5, 0) in the inequality $x - 2y \ge 0$, we get

$5 - 2 \times 0 \ge 0 = 5 \ge 0$, which is true.

So, the half plane is towards the X-axis. Since, $x, y \ge 0$ the feasible region lies in the first quadrant.



On solving equations $x - 2y = 0$ and $x + y = 60$, we get $D(40, 20)$ and solving equations $x - 2y = 0$ and $x + 2y = 120$, we get C (60, 30).

So, the feasible region is $ABCDA$. The corner points of the feasible region are A (60, 0), B (120, 0), C (60, 30) and D (40, 20). The values of Z at these points are as follows

Corner point	$Z = 5x + 10y$
A (60, 0)	300 (minimum)
B (120, 0)	600
C (60, 30)	600
D (40, 20)	400

So, the minimum value of Z is 300 at the point (60, 0).

EXAMPLE |9| Find graphically, the maximum value of $Z = 2x + 5y$, subject to constraints given below.
$2x + 4y \le 8,\ 3x + y \le 6,\ x + y \le 4,\ x \ge 0,\ y \ge 0$ **[Delhi 2015]**

Sol. We have the following LPP

Maximise $Z = 2x + 5y$

subject to constraints,

$$2x + 4y \le 8 \text{ or } x + 2y \le 4 \quad \text{...(i)}$$
$$3x + y \le 6 \quad \text{...(ii)}$$
$$x + y \le 4 \quad \text{...(iii)}$$

and $x \ge 0,\ y \ge 0$ ...(iv)

Table for line $x + 2y = 4$ is

x	4	0
y	0	2

So, the line passes through the points (4, 0) and (0, 2).

On putting (0, 0) in the inequality $x + 2y \le 4$, we get

$0 + 0 \le 4$, which is true.

So, the half plane is towards the origin.

Table for line $3x + y = 6$ is

x	2	0
y	0	6

So, the line passes through the points (2, 0) and (0, 6).

On putting (0, 0) in the inequality $3x + y \le 6$, we get

$0 + 0 \le 6$, which is true.

So, the half plane is towards the origin.

Table for line $x + y = 4$ is

x	4	0
y	0	4

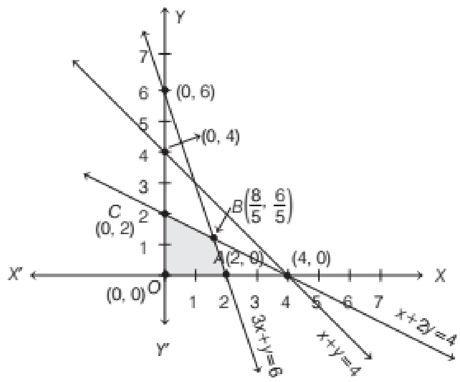
So, the line passes through the points (4, 0) and (0, 4).

On putting (0, 0) in the inequality $x + y \le 4$, we get $0 + 0 \le 4$, which is true.

So, the half plane is towards the origin.

Also, $x \ge 0$ and $y \ge 0$, so the region lies in the I quadrant. The intersection point of lines corresponding to Eqs. (i) and (ii) is $B\left(\frac{8}{5}, \frac{6}{5}\right)$.

The above inequations (i), (ii), (iii) and (iv) can be represented in the graph as shown below



Clearly, the feasible region is $OABCO$, where corner points are $O(0, 0)$, $A(2, 0)$, $B\left(\frac{8}{5}, \frac{6}{5}\right)$ and $C(0, 2)$.

The values of Z at corner points are given below

Corner points	$Z = 2x + 5y$
$O(0, 0)$	$Z = 2 \times 0 + 5 \times 0 = 0$
$A(2, 0)$	$Z = 2 \times 2 + 5 \times 0 = 4$
$B\left(\frac{8}{5}, \frac{6}{5}\right)$	$Z = 2 \times \frac{8}{5} + 5 \times \frac{6}{5} = \frac{46}{5} = 9.2$
$C(0, 2)$	$Z = 2 \times 0 + 5 \times 2 = 10$

Hence, the maximum value of Z is 10 at $C(0, 2)$.

EXAMPLE |10| Maximise and minimise $Z = x + 2y$ subject to the constraints

$$x + 2y \geq 100,$$
$$2x - y \leq 0,$$
$$2x + y \leq 200$$
and $$x, y \geq 0$$

Solve the above LPP graphically. **[All India 2017]**

Sol. Our problem is to minimise and maximise

$Z = x + 2y$ ...(i)

Subject to constraints, $x + 2y \geq 100$ ...(ii)

$2x - y \leq 0$ ...(iii)

$2x + y \leq 200$ ...(iv)

and $x \geq 0, y \geq 0$ ...(v)

Table for line $x + 2y = 100$ is

x	0	100
y	50	0

So, the line $x + 2y = 100$ is passing through the points (0, 50) and (100, 0).

On putting (0, 0) in the inequality $x + 2y \geq 100$, we get

$0 + 2 \times 0 \geq 100 \Rightarrow 0 \geq 100$, which is not true.

So, the half plane is away from the origin.

Table for line $2x - y = 0$ is

x	0	10
y	0	20

So, the line $2x - y = 0$ is passing through the points (0, 0) and (10, 20).

On putting (5, 0) in the inequality $2x - y \leq 0$, we get

$2 \times 5 - 0 \leq 0 \Rightarrow 10 \leq 0$, which is not true.

So, the half plane is towards Y-axis.

Table for line $2x + y = 200$ is

x	0	100
y	200	0

So, the line $2x + y = 200$ is passing through the points (0, 200) and (100, 0).

On putting (0, 0) in the inequality $2x + y \leq 200$, we get

$2 \times 0 + 0 \leq 200 \Rightarrow 0 \leq 200$, which is true.

So, the half plane is towards the origin.

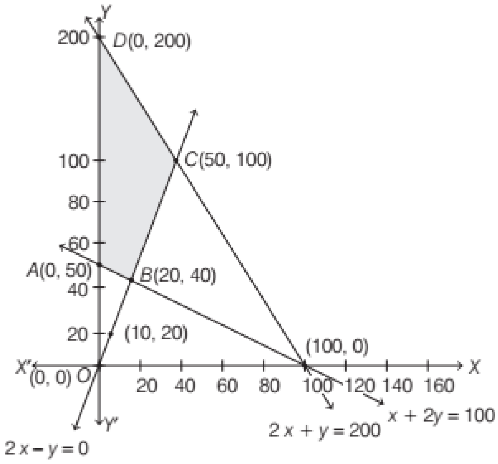
Also, $x, y \geq 0$.

So, the region lies in the I quadrant.

On solving equations $2x - y = 0$ and $x + 2y = 100$, we get $B(20, 40)$.

Again, solving the equations $2x - y = 0$ and $2x + y = 200$, we get $C(50, 100)$.

Clearly, the feasible region is $ABCDA$.



The corner points of the feasible region are $A(0, 50)$, $B(20, 40)$, $C(50, 100)$ and $D(0, 200)$.

The values of Z at corner points are given below

Corner points	$Z = x + 2y$
$A(0, 50)$	$Z = 0 + 2 \times 50 = 100$
$B(20, 40)$	$Z = 20 + 2 \times 40 = 100$
$C(50, 100)$	$Z = 50 + 2 \times 100 = 250$
$D(0, 200)$	$Z = 0 + 2 \times 200 = 400$

The maximum value of Z is 400 at $D(0, 200)$ and the minimum value of Z is 100 at all the points on the line segment joining $A(0, 50)$ and $B(20, 40)$.

EXAMPLE |11| Determine graphically the minimum value of the objective function $Z = -50x + 20y$, subject to constraints $2x - y \geq -5$, $3x + y \geq 3$, $2x - 3y \leq 12$ and $x \geq 0, y \geq 0$.

Sol. Given objective function is

Minimise $Z = -50x + 20y$

subject to constraints, $2x - y \geq -5$ ...(i)

$3x + y \geq 3$ ...(ii)

$2x - 3y \leq 12$ ...(iii)

and $x \geq 0, y \geq 0$ ...(iv)

Table for line $2x - y = -5$ is

x	$-5/2$	0
y	0	5

So, the line passes through the points $\left(\frac{-5}{2}, 0\right)$ and (0, 5).

On putting (0, 0) in the inequality $2x - y \geq -5$, we get

$0 - 0 \geq -5$

$\Rightarrow \quad 0 \geq -5$, which is true.

So, the half plane is towards the origin.

Table for line $3x + y = 3$ is

x	0	1
y	3	0

So, the line passes through the points (0, 3) and (1, 0).

On putting (0, 0) in the inequality $3x + y \geq 3$, we get

$0 + 0 \geq 3$

$\Rightarrow \quad 0 \geq 3$, which is not true.

So, the half plane is away from the origin.

Table for line $2x - 3y = 12$ is

x	0	6
y	-4	0

On putting (0, 0) in the inequality $2x - 3y \leq 12$, we get

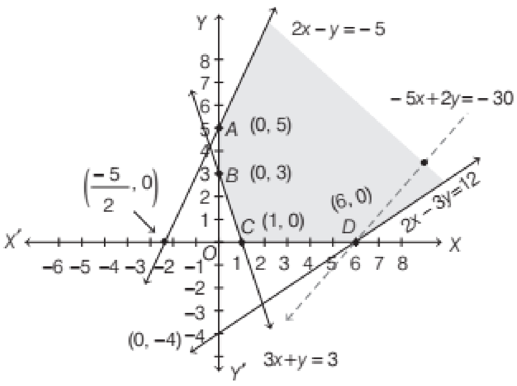
$0 - 0 \leq 12$

$\Rightarrow \quad 0 \leq 12$, which is true.

So, the half plane is towards the origin.

Also, $x \geq 0$ and $y \geq 0$, so the region lies in the I quadrant.

On drawing the graph of each linear equation, we get the following graph. In first quadrant, these equations has no intersection point.



Thus, we get the common shaded region $ABCD$, which gives the feasible region and it is unbounded.

The corner points of feasible region are $A(0, 5)$, $B(0, 3)$, $C(1, 0)$ and $D(6, 0)$.

The value of Z at corner points are given below

Corner points	$Z = -50x + 20y$
$A(0, 5)$	$Z = -50(0) + 20(5) = 100$
$B(0, 3)$	$Z = -50(0) + 20(3) = 60$
$C(1, 0)$	$Z = -50(1) + 20(0) = -50$
$D(6, 0)$	$Z = -50(6) + 20(0) = -300$

Here, feasible region is unbounded so the minimum and maximum value may or may not be exist.

Now, we draw a dotted line of inequation

$$-50x + 20y < -300 \text{ or } -5x + 2y < -30$$

Here, we see that half plane determined by $-5x + 2y < -30$ has a point in common with the feasible region.

Hence, no minimum value exists.

EXAMPLE |12| Minimise $Z = x + 2y$, subject to constraints $2x + y \geq 3$, $x + 2y \geq 6$ and $x, y \geq 0$.

(i) Show that the minimum of Z occurs at more than two points.

(ii) Also, check whether the point $P(5, 2)$ lies inside the shaded portion, if it lies inside the shaded region, then find the distance between the origin and point.

[NCERT]

Sol. Our problem is to minimise, $Z = x + 2y$ ...(i)

subject to constraints,

$2x + y \geq 3$ ...(ii)

$x + 2y \geq 6$ ...(iii)

and $x \geq 0, y \geq 0$ ...(iv)

Table for line $2x + y = 3$ is

x	0	3/2
y	3	0

So, line passes through the points (0, 3) and $\left(\frac{3}{2}, 0\right)$.

On putting (0, 0) in the inequality $2x + y \geq 3$, we get

$2 \times 0 + 0 \geq 3$

$\Rightarrow \quad 0 \geq 3$, which is not true.

So, the half plane is away from the origin.

Table for line $x + 2y = 6$ is

x	0	6
y	3	0

So, line passes through the points (0, 3) and (6, 0).

On putting (0, 0) in the inequality $x + 2y \geq 6$, we get

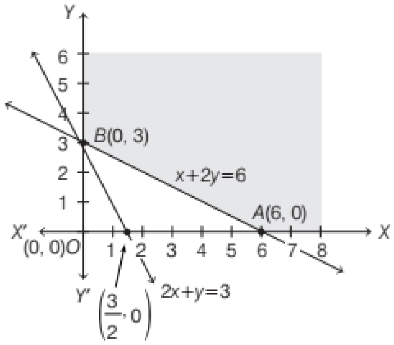
$0 + 2 \times 0 \geq 6$

$\Rightarrow \quad 0 \geq 6$, which is not true.

So, the half plane is away from the origin.

Also, $x \geq 0$ and $y \geq 0$, so the region lies in the I quadrant.

The intersection point of the lines $x + 2y = 6$ and $2x + y = 3$ is $B(0, 3)$.



The corner points of the feasible region are $A(6, 0)$ and $B(0, 3)$.

The values of Z at the corner points are given below

Corner points	$Z = x + 2y$
$A(6, 0)$	$Z = 6 + 2 \times 0 = 6$
$B(0, 3)$	$Z = 0 + 2 \times 3 = 6$

As the feasible region is unbounded, therefore 6 may or may not be the minimum value of Z. For this we draw a dotted graph of the inequality $x + 2y < 6$ and check whether the resulting half plane has points in common with a feasible region or not. It can be seen that the feasible region has no common point with $x + 2y < 6$.

Therefore, the minimum value of Z is 6.

Thus, the minimum value of Z occurs at more than 2 points. [The value of Z is minimum at every point on the line segment AB]

Let $f(x, y) = x + 2y - 6$

At point $P(5, 2)$, $f(5, 2) = 5 + 2 \times 2 - 6 = 3 > 0$

Hence, point lies inside the shaded region.

Now, distance between O and P

$$= \sqrt{(5-0)^2 + (2-0)^2}$$

$$= \sqrt{25 + 4} = \sqrt{29} = 5.39$$

EXAMPLE |13| Solve the following LPP graphically. Maximise $Z = 3x + 2y$, subject to constraints are $x + 2y \le 10$, $3x + y \le 15$ and $x \ge 0$, $y \ge 0$. Also, determine the area of the feasible region.

Sol. Our problem is to maximise $Z = 3x + 2y$ ...(i)

subject to constraints,

$$x + 2y \le 10 \quad \text{...(ii)}$$

$$3x + y \le 15 \quad \text{...(iii)}$$

and $x \ge 0, \ y \ge 0$ ...(iv)

Table for line $x + 2y = 10$ is

x	0	4
y	5	3

So, the line passes through the points (0, 5) and (4, 3).

On putting (0, 0) in the inequality $x + 2y \le 10$, we get

$$0 + 2 \times 0 \le 10$$

$\Rightarrow \quad 0 \le 10$, which is true.

So, the half plane is towards the origin.

Table for line $3x + y = 15$ is

x	4	5
y	3	0

So, the line passes through the points (4, 3) and (5, 0).

On putting (0, 0) in the inequality $3x + y \le 15$, we get

$$3 \times 0 + 0 \le 15$$

$\Rightarrow \quad 0 \le 15$, which is true.

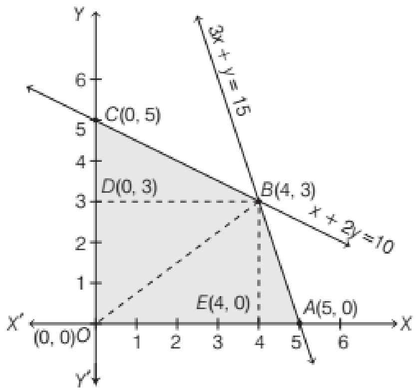
So, the half plane is towards the origin.

Also, $x, y \ge 0$, so the region lies in the I quadrant.

On solving equations $x + 2y = 10$ and $3x + y = 15$, we get $x = 4$ and $y = 3$

So, the intersection point is $B(4, 3)$.

$\therefore$ Feasible region is $OABCO$.



The corner points of the feasible region are $O(0, 0)$, $A(5, 0)$, $B(4, 3)$ and $C(0, 5)$.

The values of Z at the corner points are given below

Corner points	$Z = 3x + 2y$
$O(0, 0)$	$Z = 3 \times 0 + 2 \times 0 = 0$
$A(5, 0)$	$Z = 3 \times 5 + 2 \times 0 = 15$
$B(4, 3)$	$Z = 3 \times 4 + 2 \times 3 = 18$
$C(0, 5)$	$Z = 3 \times 0 + 2 \times 5 = 10$

Therefore, the maximum value of Z is 18 at the point $B(4, 3)$.

$\therefore$ Area of feasible region

$$= \text{Area of } \Delta BOC + \text{Area of } \Delta OAB$$

$$= \frac{1}{2} \times OC \times BD + \frac{1}{2} \times OA \times BE$$

$$= \frac{1}{2} \times 5 \times 4 + \frac{1}{2} \times 5 \times 3$$

$$= 10 + \frac{15}{2} = 10 + 7.5 = 17.5 \text{ sq units}$$

EXAMPLE |14| The matrix inequality is shown below

$$\begin{bmatrix} x+3y \\ x+y \end{bmatrix} \geq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(i) Make a linear inequation from the above inequality of matrices.

(ii) Find the minimise $Z = 3x + 5y$, subject to the constraints as above linear inequations and $x, y \geq 0$.

Sol. (i) Our problem is to minimise $Z = 3x + 5y$

subject to constraints, $x + 3y \geq 3$

$x + y \geq 2$

and $x \geq 0,\ y \geq 0$

(ii) Table for line $x + 3y = 3$ is

x	0	3
y	1	0

So, the line passes through the points (0, 1) and (3, 0).

On putting (0, 0) in the inequality $x + 3y \geq 3$, we get

$0 + 3 \times 0 \geq 3$

$\Rightarrow \quad 0 \geq 3$, which is not true.

So, the half plane is away from the origin.

Also, $x, y \geq 0$, so the feasible region lies in the I quadrant.

Table for line $x + y = 2$ is

x	0	2
y	2	0

So, the line passes through the points (0, 2) and (2, 0).

On putting (0, 0) in the inequality $x + y \geq 2$, we get

$0 + 0 \geq 2 \Rightarrow 0 \geq 2$,

which is not true.

So, the half plane is away from the origin.

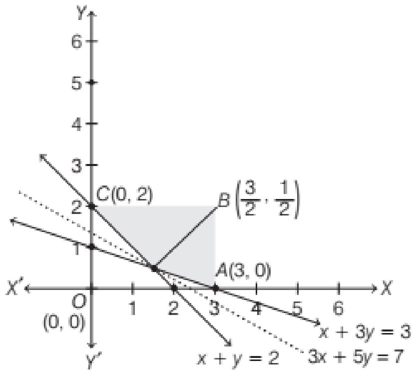
It can be seen that the feasible region is unbounded.

On solving equations $x + y = 2$ and $x + 3y = 3$, we get

$$x = \frac{3}{2} \text{ and } y = \frac{1}{2}$$

$\therefore$ Intersection point is $B\left(\frac{3}{2}, \frac{1}{2}\right)$.

The corner points of the feasible region are $A(3, 0)$, $B\left(\frac{3}{2}, \frac{1}{2}\right)$ and $C(0, 2)$.



The values of Z at the corner points are given below

Corner points	$Z = 3x + 5y$
$A(3, 0)$	$Z = 3 \times 3 + 5 \times 0 = 9$
$B\left(\frac{3}{2}, \frac{1}{2}\right)$	$Z = 3 \times \frac{3}{2} + 5 \times \frac{1}{2} = 7$
$C(0, 2)$	$Z = 3 \times 0 + 5 \times 2 = 10$

As the feasible region is unbounded, therefore 7 may or may not be the minimum value of Z.

For this, we draw the graph of the inequality, $3x + 5y < 7$ and check whether the resulting half plane has points in common with the feasible region or not.

It can be seen that, the feasible region has no common point with $3x + 5y < 7$.

Therefore, the minimum value of Z is 7 at $B\left(\frac{3}{2}, \frac{1}{2}\right)$.

SUMMARY

- **Linear Programming Problem** An LLP is one that is concerned with finding the optimal value (maximum or minimum value) of a linear function of several variables, subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities.
- **Mathematical Form of LPP** The general mathematical form of a linear programming problem may be written as
Maximise or Minimise $Z = c_1x + c_2y$
subject to constraints are $a_1x + b_1y \le d_1$, $a_2x + b_2y \le d_2$, etc.
and non-negative restrictions are $x \ge 0, y \ge 0$.
- **Important Terms Related to LPP**
  - (i) The linear inequations or inequalities or restrictions on the variables of a linear programming problem are called **constraints**. The conditions $x \ge 0, y \ge 0$ are called **non-negative restrictions**.
  - (ii) A problem which seeks to maximise or minimise a linear function subject to certain constraints as determined by a set of linear inequalities is called an **optimisation problem**.
  - (iii) A linear function of two or more variables which has to be maximised or minimised under the given restrictions is called an objective function. The variables used in the objective function are called **decision variables**.
  - (iv) The maximum or minimum value of an objective function is known as the **optimal value** of LPP.
  - (v) The common region determined by all the constraints including non-negative constraints $x, y \ge 0$ of a linear programming problem is called the **feasible region** or **solution region**. The region other than feasible region is called an **infeasible region**.
  - (vi) A feasible region of a system of linear inequalities is said to be **bounded**, if it can be enclosed within a circle. Otherwise, it is said to be **unbounded region**, i.e. the feasible region does extend indefinitely in any direction.
  - (vii) Points within and on the boundary of the feasible region represent feasible solution of the constraints.
  - (viii) A feasible solution at which the objective function has optimal value (maximum or minimum) is called the **optimal solution** or **optimal feasible solution** of the linear programming problem.
  - (ix) The process of obtaining the optimal solution of the linear programming problem is called **optimisation technique**.
- **Important Theorems**
  - (i) **Theorem 1** Let R be the feasible region (convex polygon) for a linear programming problem and $Z = ax + by$ be the objective function.

    When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

    (A corner point of a feasible region is a point of intersection of two boundary lines in the region).
  - (ii) **Theorem 2** Let R be the feasible region for a linear programming problem and $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.
- **Graphical Method** (or Corner Point Method) **for Solving LPP**
  - (i) Find the feasible region of the LPP and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
  - (ii) Find the value of objective function $Z = ax + by$ at each corner point. Let M and m respectively denote the largest and the smallest values at these points.
    - (a) When the feasible region is bounded, M and m are the maximum and minimum values of Z.
    - (b) When the feasible region is unbounded, then
      - M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.
      - m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

|1 Mark|

1 The graph of the inequality $2x + 3y > 6$ is **[All India 2020]**

(a) half plane that contains the origin

(b) half plane that neither contains the origin nor the points of the line $2x + 3y = 6$

(c) whole XOY-plane excluding the points on the line $2x + 3y = 6$

(d) entire XOY-plane

2 In an LPP, if the objective function has $z = ax + by$ has the same maximum value on two corner points of the feasible region, then the number of points at which z_{max} occurs is **[Delhi 2020]**

(a) 0

(b) 2

(c) finite

(d) infinite

3 The optimal value of the objective function is attained at the point is

(a) given by intersection of inequations with axes only

(b) given by intersection of inequations with X-axis only

(c) given by corner points of the feasible region

(d) None of the above

4 The linear programming problem minimize $Z = 3x + 2y$ subject to constraints $x + y \geq 8$, $3x + 5y \leq 15$, $x \geq 0$ and $y \geq 0$, has

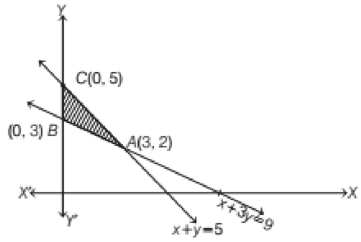
(a) one solution

(b) no feasible solution

(c) two solutions

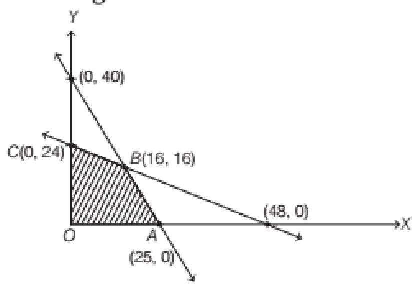
(d) infinitely many solutions

5 The feasible region for an LPP is shown in the following figure. Then, the minimum value of $Z = 11x + 7y$ is



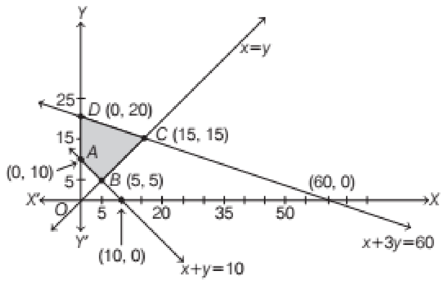
(a) 21 (b) 47 (c) 20 (d) 31

6 The maximum value of $Z = 4x + 3y$, if the feasible region for an LPP is as shown below, is



(a) 112 (b) 100 (c) 72 (d) 110

7 Based on the given shaded region as the feasible region in the graph, at which point (s) is the objective function $Z = 3x + 9y$ maximum.



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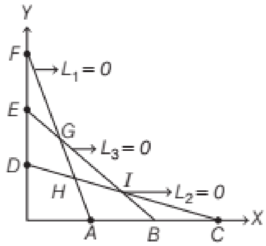
(a) Point B

(b) Point C

(c) Point D

(d) Every point on the line segment CD

8 The feasible region for the following constraints $L_1 \leq 0, L_2 \geq 0, L_3 = 0, x \geq 0, y \geq 0$ in the diagram shown is



(a) area DHF

(b) area AHC

(c) line segment EG

(d) line segment GI

9 The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5), (15, 15), (0, 20). Let $Z = px + qy$, where $p, q > 0$. Then, the condition on p and q, so that the maximum of Z occurs at both the points (15, 15) and (0, 20) is **[NCERT Exemplar]**

(a) $p = q$

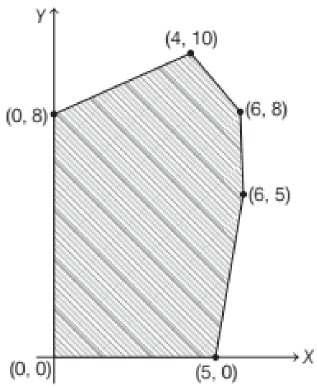
(b) $p = 2q$

(c) $q = 2p$

(d) $q = 3p$

10. In the given graph, the feasible region for a LPP is shaded. The objective function $Z = 2x - 3y$ will be minimum at

[Latest CBSE Sample Paper 2021 (Term I)]



(a) (4, 10) (b) (6, 8)

(c) (0, 8) (d) (6, 5)

SHORT ANSWER Type II Questions

|4 Marks |

Solve the following linear programming problems graphically.

11 Maximise $Z = 6x + 11y$

subject to constraints $2x + y \leq 104, x + 2y \leq 76$

and $x \geq 0, y \geq 0$

Directions (Q. Nos. 12-14) *Solve the following LPP graphically.*

12 Maximise $Z = 3x + 4y$, subject to constraints are $x + y \leq 1; x \geq 0, y \geq 0.$ **[NCERT Exemplar]**

13 Minimise $Z = 2x + 4y$, subject to constraints are $x + y \geq 8, x + 4y \geq 12, x \geq 3, y \geq 2$ and $x, y \geq 0.$

14 Maximise and minimise $Z = 3x - 4y$, subject to constraints are $x - 2y \leq 0, -3x + y \leq 4, x - y \leq 6$ and $x, y \geq 0.$ **[NCERT Exemplar]**

15 Minimise $Z = 6x + 3y$

subject to the constraints

$4x + y \geq 80, x + 5y \geq 115, 3x + 2y \leq 150, x \geq 0, y \geq 0.$

[Delhi 2017C]

16 Maximise $Z = 8000x + 12000y$

subject to the constraints

$3x + 4y \leq 60, x + 3y \leq 30$ and $x \geq 0, y \geq 0$

[Delhi 2017C]

17 Solve maximise $Z = 105x + 90y$ graphically

under the following constraints.

$x + y \leq 50, 2x + y \leq 80, x \geq 20$ and $x \geq 0, y \geq 0$

[Delhi 2017C]

18 Solve the following linear programming problem graphically.

Maximise $Z = 4x + y$

subject to the constraints

$x + y \leq 50, 3x + y \leq 90, x \geq 10; x, y \geq 0.$ **[Delhi 2017]**

19 Solve the following linear programming problem graphically.

Maximise $Z = 20x + 10y$

subject to the following constraints

$x + 2y \leq 28, 3x + y \leq 24, x \geq 2; x, y \geq 0$ **[Delhi 2017]**

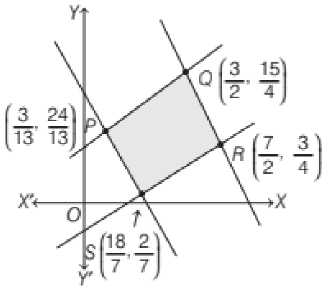
20 Solve the following linear programming problem graphically.

Maximise $Z = 7x + 10y$

subject to the constraints

$$4x + 6y \le 240,\ 6x + 3y \le 240,\ x \ge 10 \text{ and } x \ge 0,\ y \ge 0$$ **[All India 2017]**

21 In the given figure, the feasible region for a LPP is shown. Determine the maximum and minimum value of $Z = x + 2y$. **[NCERT Exemplar]**



22 The corner points of the feasible region determined by the system of linear constraints are (0, 3), (1, 1) and (3, 0). Let $Z = px + qy$, where $p, q > 0$. Find the condition in p and q, so that the minimum of Z occurs at (3, 0) and (1, 1). **[NCERT Exemplar]**

| ANSWERS |

1. (b) **2.** (d) **3.** (c) **4.** (b) **5.** (a) **6.** (a) **7.** (d) **8.** (c)
9. (d) **10.** (c)

11. **Hint** Do same as Example 5. **Ans.** Maximum $Z = 440$ at (44, 16)

12. **Hint** Do same as Example 5. **Ans.** Maximum value of Z is 4 at (0, 1).

13. **Hint** Do same as Example 6. **Ans.** Minimum value of $Z = 16$ at $x = 4,\ y = 2$

14. **Hint** Do same as Example 5. **Ans.** Maximum value = 12 and no minimum value exist.

15. **Hint** Do same as Example 6. **Ans.** Minimum $Z = 150$, when $x = 15,\ y = 20$.

16. **Hint** Do same as Example 5. **Ans.** Maximum $Z = 168000$, when $x = 12,\ y = 6$.

17. **Hint** Do same as Example 5. **Ans.** Maximum value of Z is 4950, when $x = 30,\ y = 20$.

18. **Hint** Do same as Example 5. **Ans.** Maximum $Z = 120$ at $(30,\ 0)$

19. **Hint** Do same as Example 5. **Ans.** Maximum $Z = 200$ at $(4, 12)$

20. **Hint** Do same as Example 5. **Ans.** Maximum $Z = 410$ at $(30,\ 20)$

21. **Hint** Do same as Example 5. **Ans.** Maximum and minimum values of Z are 9 and 3.14

22. $2p = q$